Electoral Systems, Legislative Process, and Income Taxation

YAN CHEN
The University of Michigan

Abstract

We characterize the equilibrium income tax schedules and the optimality conditions under two types of political institutions, a two-party plurality system with a single district, and one with multiple districts where tax policies are determined through a legislature. It is shown that the exogenous social welfare functions in the optimal taxation literature can be endogenously determined by explicitly modeling the political institutions, which put different welfare weights on different subsets of the population. This paper also extends the Coughlin probabilistic voting model and the Baron–Ferejohn legislative bargaining model to a function space.

1. Introduction

Fiscal policies, such as taxation and public goods provision, are determined, in reality, by various political institutions rather than by a benevolent social planner. In 27 American states laws can be proposed only by elected representatives; in the other 23 states, laws can be initiated and approved by popular vote. Empirical studies have found significant fiscal effects resulting from these different political institutions. For example, Matsusaka (1995) found that overall spending is significantly higher in pure representative states. However, local spending is higher and state spending is lower in initiative states. In these empirical studies time series

Department of Economics, The University of Michigan, Ann Arbor, MI 48109-1220 (yanchen@umich.edu).

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data from various states, possibly with different legislative processes, are pooled to get the aggregate effects.

In this paper we construct a theoretical model of different political institutions in order to understand how political institutions affect fiscal policies. Compared to empirical studies, theoretical analysis allows us to explicitly model and simulate details of legislative processes and to understand why different institutions result in different fiscal policies.

Traditionally, the optimal income taxation literature, starting from Mirrlees (1971), studies the features of income tax schedules, which arise when a social planner maximizes an exogenously given social welfare function, subject to incentive compatibility constraints and an exogenously given revenue requirement. These models recognize the incentive effects of income taxation. In the analysis, most of them start with unrestricted tax schedules, without a priori limitations. The main shortcoming of these models is the neglect of institutional constraints. The social welfare function is not derived explicitly. Therefore, one has no reason to believe that any particular social welfare function captures the political economy of real policy choices.

We model income taxation and public goods provision as the outcome of political processes. We compare the equilibrium tax schedules of two different kinds of institutions: a two-party plurality system with a single district, and one with multiple districts where the tax policy is determined through a legislature. Traditionally, the Achilles' heel has been getting equilibrium in a multidimensional model of political equilibrium (McKelvey 1979). Therefore, previous research in this area (e.g., Roberts 1977; Snyder and Kramer 1988; Berliant and Gouveia 1991) either starts with a restricted set of tax schedules, such as a linear tax, or puts restrictions on the environment. The only institution studied previously is simple majority rule, and nearly all results focus on the median voter.

This paper extends the probabilistic voting model and the Baron–Ferejohn (1989) legislative bargaining model to a function space, so we only need minimal restriction on the class of tax schedules. Furthermore, two types of political institutions are studied: a two-party plurality system under a single district (simple majority rule), and a two-party plurality system under multiple districts with a legislature deciding the final policy outcome.

By explicitly modeling the political institutions, we can characterize the equilibrium tax schedules and conditions under which they are optimal, and thereby endogenously determine the social welfare function. Under plurality rule, the equilibrium tax schedule of two-candidate, single-district competition is compared with the equilibrium outcome from a legislative process when there are multiple districts. We establish that each equilibrium is equivalent to an optimal tax schedule for some social welfare weights. Furthermore, the equilibrium that arises in a two-party, single-district competition puts equal welfare weight over the whole population, while the equilibrium tax schedule of the legislative process puts more weight on those subsets of the population whose legislators are in the majority coalition.

In Section 2 we construct a general equilibrium model where the level of public good is endogenously determined. Section 3 extends Coughlin's probabilistic voting model to a function space and uses it to characterize the equilibrium income tax schedules under a two-party plurality system for a single district, and under a stochastic legislative game when there are multiple districts. Optimality conditions for these equilibria are also derived, thus establishing the relationship between these positive models and traditional optimal income taxation models. Section 4 presents a numerical example. Section 5 concludes the paper. Proofs not given in the sections are gathered in the appendixes.

2. The Model

A general equilibrium model is constructed in which the amount of public good is endogenously determined. The general problem analyzed in this section uses a framework similar to that of Mirrlees (1971), but includes a public good, financed by the tax revenue instead of having an exogenous revenue requirement. This model serves as a building block for the introduction of political institutions. It turns out that the equilibrium outcomes of the two political institutions we consider will be special cases of the optimal income tax model, in the sense that the equilibrium tax schedules from political processes are as if some social welfare functions are maximized. Therefore, the equilibrium outcomes of the political processes correspond to two points on the second-best frontier.

Suppose individuals are identified by a single parameter, $\omega \in \Omega = [\underline{\omega}, \bar{\omega}] \subset R_{++}$, which can be interpreted as the wage rate or ability level of an individual. Assume that $\omega \sim F(\cdot)$, and that ω has a density function $f(\omega)$, and $f(\omega) > 0$ a.s. on Ω. Call an individual whose ability-parameter is ω an ω-person. The individual parameter, ω, is private information, but its distribution is common knowledge. There are three commodities: a consumption good, $x \in R_+$, labor, $l \in [0,1]$, and a public good, $y \in R_+$. We normalize the endowment of time to 1. The utility function, $u(x, l, y)$ satisfies the following assumptions.

Assumption 1: *$u(x, l, y) = x + v(l, y)$, where $v(\cdot,\cdot)$ is concave, C^3, $u_2 = v_1 < 0$, $u_3 = v_2 > 0$, $u_{23} = 0$, and satisfies the Inada conditions: $\lim_{l \to 1} u_2(x, l, y) = -\infty$; $\lim_{y \to 0} u_3(x, l, y) = \infty$.*

As usual, the Inada conditions are introduced to guarantee interior solutions. Additive separability is not essential for the main results, but will greatly simplify the algebra.

Assumption 2: *The marginal utility of leisure is convex: $u_{222} \leq 0$.*

Assumption 2 is introduced so that the second-order condition is satisfied when we later use the first-order approach to solve the optimal taxation problem. One example that satisfies Assumptions 1 and 2 is $u(x, l, y) = x + \beta \ln(1 - l) + (1 - \beta) \ln y$.

Let $I(\omega) = \omega l$ be the income of the ω-person, which is observable by the government. Then $I : \Omega \to R_+$ is defined as the *income function,* and $T : R_+ \to R$ is defined as the *income tax function.* Define a *revenue requirement function,* $\tau: \Omega \to R$. The revenue requirement function satisfies the following assumption.

ASSUMPTION 3: *The revenue requirement function, $\tau : \Omega \to R$, is lower semicontinuous and bounded; that is, $\omega > \tau(\omega) > -\infty$.*

The upper bound of the revenue requirement function says that you cannot tax a person more than the maximum she could possibly earn. The lower bound requires that the government cannot subsidize any individual an infinite amount, which will be used to establish the compactness of the policy space.

There have been two different approaches to analyze the optimal income taxation problem. The first approach (e.g., in Atkinson and Stiglitz 1980) lets the government set the income tax function and lets individuals choose labor supply. Although this approach is intuitive, it is not user friendly. The Hamiltonian is written in an unnatural way with the level of utility being a state variable and the level of labor supply being a control variable. This requires rewriting the problem in such a way that consumption is a function of the level of utility and labor. The second approach, which is equivalent to the first one, uses the differentiable approach to the revelation principle.[1] It transforms the problem of taxation of income (the indirect mechanism) into the direct mechanism: An agent reports his type, ω, based on which he is required to supply labor, $l(\omega)$, and pay taxes, $\tau(\omega)$. The government wants to find a tax function T that implements τ in the sense that $T(I(\omega; T)) = \tau(\omega)$. We use the second approach in this paper because it is much easier to solve the control problem and to check the second-order condition. Lemma 1 shows that the two approaches are equivalent.

LEMMA 1: *The two approaches, (1) the government setting income tax and agents choosing labor supply, and (2) agents reporting types, based on which the government specifies labor supply and taxes, yield the same incentive compatibility constraint.*

When we use the first-order approach to study the optimal taxation problem, it is necessary to check the second-order condition of the incentive compatibility constraint because this type of model often implies

[1] I thank Miguel Gouveia for pointing out the application of this approach to the optimal taxation problem. See Berliant and Gouveia (1994).

bunching of agents. The following lemma states the familiar monotonicity condition, which is one of the two sufficient conditions to rule out the bunching of agents.

LEMMA 2 (Monotonicity): *If $l(\cdot)$ is increasing, then there exists a revenue requirement function, $\tau(\cdot)$ such that ($l(\cdot)$ and $\tau(\cdot)$) are truthfully implementable in dominant strategies.*

It is straightforward that if $l(\cdot)$ is increasing then $I(\cdot)$ is nondecreasing. In other words, in equilibrium, after all behavioral adjustments, income must be a nondecreasing function of ability.

Given a revenue requirement function, $\tau(\omega)$, and a labor supply function, $l(\omega)$, an ω-person chooses to report his type, ω', to maximize his utility, $\max_{\omega'} u(\omega l(\omega') - \tau(\omega'), l(\omega'), y)$. From the proof of Lemma 1 the incentive compatibility constraint is $\omega l'(\omega) - \tau'(\omega) + u_2 l'(\omega) = 0$. Lemma 2 and Assumption 2 guarantee that the second-order condition for the incentive compatibility constraint is satisfied. Solving this problem gives us an individual's optimal reported type, ω, and, thus, his optimal labor supply, $l(\omega)$, his optimal amount of income, $I(\omega)$, and his private good consumption, $x(\omega) = I(\omega) - \tau(\omega)$. The total supply of labor adjusted for quality is $L^s = \int_\Omega \omega l(\omega)\, dF(\omega)$, the aggregate demand for the private good is $X^d = \int_\Omega x(\omega)\, dF(\omega)$, and the total tax revenue is $\int_\Omega \tau(\omega)\, dF(\omega)$.

On the production side, assume that firms are price-takers. The input for the production of the private good is labor, which, adjusted for quality, equals $L^d = \int_\Omega \omega l(\omega)\, dF(\omega)$. The public good is produced from the private good. Assume that all firms are identical and that they maximize profit by choosing the optimal amount of labor input in the production of the private good and the public good. The production functions of the private good and the public good are assumed to be linear. The total amount of private good produced is $X^T = aL$, and the total amount of public good produced from the private good is $y = b(X^T - X)$, where X is the total private good consumption. Normalize the price of the private good to 1, and let the price of the public good be p. In equilibrium, the firms' profit is zero, and demand equals supply in all markets. So we have $\pi = X^s + py - L^d = 0$. With linear technology, $X^T = aL^d$ and $y = b(X^T - X^s)$. Therefore,

$$X + py = L^d = \frac{X^T}{a} = \left(\frac{y}{b} + X^s\right)/a,$$

or

$$a(X^s + py) = X^s + \frac{y}{b}.$$

Hence, $a = 1$ and $p = \frac{1}{b}$.

The government uses the tax revenue to purchase the public good. Therefore, the balanced budget constraint is $py = \int_\Omega \tau(\omega)\, dF(\omega)$, or $y = b\int_\Omega \tau(\omega)\, dF(\omega)$.

The optimal income tax problem is thus defined as

$$\max_{\tau,\, l} \int_\Omega A\left(u(\omega l(\omega) - \tau(\omega), l(\omega), b\int_\Omega \tau(\omega)\; dF(\omega)\right) dF(\omega)$$

$$\text{s.t. } \omega l'(\omega) - \tau'(\omega) + u_2\, l'(\omega) = 0 \tag{IC}$$

$$l'(\omega) > 0 \tag{M}$$

$$l(\bar{\omega}) = \bar{l},\ (\bar{l}\,\text{free}) \tag{B1}$$

$$l(\underline{\omega}) = \underline{l},\ (\underline{l}\,\text{free}) \tag{B2}$$

$$\tau(\bar{\omega}) = \bar{\tau},\ (\bar{\tau}\text{ free}) \tag{B3}$$

$$\tau(\underline{\omega}) = \underline{\tau},\ (\underline{\tau}\text{ free}), \tag{B4}$$

where $A(u(\omega))$ is some exogenously given, strictly increasing, concave and differentiable welfare function. Equation (IC) is the incentive compatibility constraint; equation (M) is the monotonicity constraint; and the last four constraints, (B1) to (B4), are the boundary conditions that will be used to derive the transversality conditions.

PROPOSITION 1: *The optimal tax schedule, $T(I)$, satisfies equation (IC) and*

$$T' = \frac{\left[\int_{\underline{\omega}}^{\omega} A'\; dF - bF(\omega)\int_\Omega A' u_3\; dF(\omega)\right](1 + u_{22}\, l')}{wbf\int_\Omega A' u_3\; dF(\omega)}. \tag{1}$$

Our result is different from that of Mirrlees (1971) due to the endogeneity of the public good. We can get some immediate characterizations of the optimal income tax schedule in the following corollary.

COROLLARY 1: *The marginal tax rate for the lowest type is zero, and the marginal tax rate for the highest type is generically nonzero unless $bu_3(\bar{\omega}) = 1$ or $[1 + u_{22}\, l']_{\omega=\bar{\omega}} = 0$:*

$$T'(\underline{\omega}) = 0; \qquad T'(\bar{\omega}) = 0 \quad \text{if } bu_3(\bar{\omega}) = 1 \text{ or } [1 + u_{22}\, l']_{\omega=\bar{\omega}} = 0.$$

Proof: Plug $\underline{\omega}$ and $\bar{\omega}$, respectively, into equation (1). The results follow immediately. ■

Note that although the well-known result on the marginal tax rate of the lowest type remains true, the marginal tax rate of the highest type is generically nonzero. This result is consistent with Brito and Oakland (1977). The marginal tax rate depends on the distribution of types, the production technology of public goods, and people's preference for public goods, as well as the exact form of the social welfare function, $A(\cdot)$. In Section 3 we will demonstrate that specific forms of the social welfare function are determined by the political institutions.

3. Political Institutions and Equilibrium Income Tax Schedule

In this section we analyze how two different political institutions endogenously determine the income tax schedule as well as the social welfare functions. We first review some voting models in the literature.

3.1 The Probabilistic Voting Model and an Extension

The most difficult part of studying the equilibrium income tax schedules under various political institutions is determining the existence of equilibrium when the issue space exceeds one dimension. The *deterministic Downsian model,* which assumes complete information and no uncertainty, typically provides no predictions when there are two or more dimensions to the policy space. This difficulty in obtaining existence of an equilibrium is created by the discontinuity of voters' behavior. If candidate A offers the voter an ϵ increment of utility over candidate B, then she will switch her vote to candidate A. One way to smooth out this discontinuity is to introduce uncertainty into voters' decision processes, which might also be a descriptively more accurate representation of the real decision processes.

One approach in Ledyard (1984) uses the *Bayesian voting model,* where voters' types and their costs of voting are private information. Abstention is allowed. In the resulting equilibrium, both candidates adopt the same platform that maximizes a Samuelson–Bergson social welfare function and nobody votes. The analysis is based on an individual being pivotal in an election, which is not applicable when there is a continuum of voters/consumers.

An alternative way of modeling voting under uncertainty is the *probabilistic voting model,*[2] which can be understood as reflecting candidates' uncertainty about whom the individual voters will vote for. It uses standard statistical models for discrete choice in a game theoretic setting. We use this approach to analyze voting over income taxation and public goods provision. We will briefly describe the underlying rationale for this approach, and then extend the results to a function space.

Consider an electorate where everyone votes. In the two-candidate case, this means that the probability with which an individual ω chooses candidate i, $P^i(T_1,T_2,\omega)$, given i's platform, T_i, satisfies $P^1(T_1,T_2,\omega) + P^2(T_1,T_2,\omega) = 1$. The ω-person's utility from candidate i's platform is $\mu(T_i,\omega) = u(T_i,\omega)\exp(\epsilon_i), i = 1,2,$ where ϵ_i is a latent variable – that is,

[2] For a comprehensive treatment of this subject, see Coughlin (1992).

some characteristics of the voter not observed by the candidates. Assuming that the error term, ϵ_i, is distributed logistically and that the ω-person maximizes $\mu(T_i, \omega)$, we get the individual choice probabilities on any pair of platforms as

$$P^i(T_1, T_2, \omega) = \frac{u(T_i, \omega)}{u(T_1, \omega) + u(T_2, \omega)}.$$

This approach draws on the multinomial logit framework commonly used in econometric models of discrete choice (e.g., see Amemiya 1985, Chap. 9). Notice that the only difference between the probabilistic voting model and the logit model is how the error term enters the utility function: it enters additively in the logit model and multiplicatively in the probabilistic voting models.

Therefore, a candidate's expected vote equals

$$Ev_i(T_i | T_{-i}) = \int_\Omega \frac{u(T_i, \omega)}{u(T_1, \omega) + u(T_2, \omega)} \, dF(\omega).$$

Assume that each candidate's objective function is to maximize expected plurality, which is equivalent to maximizing the probability of winning in a large electorate. Define the expected plurality for candidate 1 as

$$EPl_1 = Ev_1 - Ev_2 = \int_\Omega \frac{u(T_1, \omega) - u(T_2, \omega)}{u(T_1, \omega) + u(T_2, \omega)} \, dF(\omega),$$

and the expected plurality for candidate 2 as $EPl_2 = -EPl_1$.

Coughlin and Nitzan (1981) characterized an equilibrium when the policy set lies in a Euclidean space.

THEOREM 1 (Coughlin 1992, Thm. 6.3): *If the policy space $X \subset R^m$ is compact, if voters vote probabilistically, and if $u(T)$ is concave in T, then an alternative, $T^* \in X \subset R^m$, is an outcome of the two-candidate electoral competition if and only if $T^* \in \operatorname{argmax} \int_\Omega \ln u(T, \omega) \, dF(\omega)$.*

Therefore, in a two-candidate competition under plurality rule, the equilibrium policy outcome is the maximand of the Nash social welfare function, $A = \int_\Omega \ln u(T, \omega) \, dF(\omega)$.

There have been some criticisms of the probabilistic voting model, mostly stemming from the assumption on the error terms in individual decision making. To construct a discrete choice model that will produce predictions consistent with the underlying theory, any proper, continuous probability distribution defined over the real line will suffice.[3] The logistic distribution is chosen because of its mathematical simplicity and its resemblance to the normal distribution. Note that the existence of major-

[3] See, for example, Green (1990, pp. 662–666).

ity rule equilibrium in multidimensional policy space is robust to specific assumptions on the distribution of errors, although the optimality condition in Theorem 1 might be sensitive to these assumptions. How the optimality condition might change if another probability distribution is assumed needs to be worked out, but that is beyond the scope of this paper.

Although, like most other voting models, experimental or empirical testings of the probabilistic voting model itself have not been performed, we can gain some confidence from the laboratory performance of a very similar type of model, the Quantal Response Equilibrium model (McKelvey and Palfrey 1995). For a logistic specification of the error structure as well, the Quantal Response Equilibrium model fits a variety of experimental datasets fairly successfully.

To study the equilibrium tax structure, we need to extend Theorem 1 to cover the case in which the policy belongs to a function space. Lemmas 3, 4, and 5 establish the compactness of the policy space.

LEMMA 3: *After tax consumption, $x(\omega, \omega_t)$, is nondecreasing in ω, where ω_t is his true type, and ω is his reported type.*

Lemma 3 is used to put more structure on the revenue requirement function. It will be used in the proof of Lemma 4.

LEMMA 4: *$\tau(\omega)$ is of bounded variation.*

Let $BV[a,b]$ denote the space of functions of bounded variation on $[a,b]$. Define $X_\tau = \{\tau : \text{lower semicontinuous and of } BV\}$, and $X_l = \{l : \text{increasing}\}$. The policy space is therefore $X = \{(l \in X_l, \tau \in X_\tau) : I.C.\}$, where $I.C.$ is the incentive compatibility constraint. To prove the existence of the electoral equilibrium, we need to show that X is compact.

LEMMA 5: *The policy space X is compact.*

Having established the compactness of the policy space, we proceed to prove that Theorem 1 can be extended to a function space.

COROLLARY 2: *In the policy space X, if voters vote probabilistically, and if $u(\cdot)$ is concave in (l, τ), then an equilibrium of the two-party electoral competition exists; furthermore, (l^*, τ^*) is an equilibrium to the electoral competition if and only if*

$$l^*, \tau^* \in \operatorname{argmax} \int_\Omega \ln u(\omega l - \tau, l, y)\, dF(\omega)$$

$$\text{s.t.} \quad \omega l'(\omega) - \tau'(\omega) + u_2\, l'(\omega) = 0, \qquad \text{(IC)}$$

$$l'(\omega) > 0, \qquad \text{(M)}$$

and the four boundary constraints, (B1)–(B4), hold.

Note that in our setting the concavity of the indirect utility function in the policy proposal, (l, τ), is guaranteed by Assumption 2.

Corollary 2 extends Theorem 1 to a function space. It establishes that the equilibrium tax schedule under a two-party plurality system with a single district can be obtained as if we are solving an optimal taxation problem, with the previously exogenously given social welfare function taking the form of the Nash social welfare function.

Next, we proceed to characterize the equilibria of two political institutions and the optimality conditions of these equilibria, which suggest that they are special cases of the optimal taxation model. The first institution is a two-party plurality system under a single district, which can be viewed as a simplified version of implementing the platform from a presidential election or the outcome of a simple majority rule/referendum. The second institution is a legislative process under a two-party plurality system with multiple districts.

3.2 Two-Party Plurality System under a Single District

From Corollary 2, the equilibrium tax schedule for a two-party plurality system under a single district is the solution to the following optimization problem:

$$\max_{\tau, l} \int_{\Omega} \ln u\left(\omega l(\omega) - \tau(\omega), l(\omega), b\int_{\Omega} \tau(\omega)\, dF(\omega)\right) dF(\omega)$$

$$\text{s.t.} \quad \omega l'(\omega) - \tau'(\omega) + u_2\, l'(\omega) = 0, \tag{IC}$$

$$l'(\omega) > 0, \tag{M}$$

and the four boundary constraints, (B1)–(B4), hold.

Solving the above problem, we get the following proposition.

PROPOSITION 2:

(a) *The equilibrium tax schedule under the single district, two-party plurality system satisfies (IC), (M), and the following equation:*

$$T' = \frac{\left[\int_{\underline{\omega}}^{\omega} 1/u\, dF - bF(\omega) \int_{\Omega} u_3/u\, dF(\omega) \right] (1 + u_{22}\, l')}{wbf \int_{\Omega} u_3/u\, dF(\omega)}.$$

(b) *It is optimal if the welfare function is* $\int_{\Omega} A(u)\, dF(\omega) = \int_{\Omega} \ln u\, dF(\omega)$.

Proof: Substituting $\ln u$ for $A(u)$ in equation(1), we obtain the above result. ■

Proposition 2 can be interpreted either as the equilibrium outcome of a two-party competition under a single district, or as the outcome of a national election, where the winning party/candidate implements his platform. A more frequently used, and also more complicated, political institution in determining public policies involves a legislature where each legislator is elected by plurality rule, and the final policy is the result of a legislative bargaining game.

3.3 Multiple Districts – Legislative Process

An alternative mechanism for deciding income tax schedules under a two-party plurality rule system is through the election of a legislative body. We model the entire process as a two-stage game. In the first stage, voters in each legislative district vote for a legislator, whose objective function is to maximize the probability of getting reelected. This voting game determines each legislator's equilibrium platform. In the second stage, the legislators, each with induced preferences, bargain to select an income tax schedule.

In the first stage, suppose voters are sophisticated in the sense that they know their legislator is not going to be a dictator in the legislature, and that the policy outcome is through a complicated process according to some legislative rule, $\gamma(\cdot):\{(l_j,\tau_j)\}_{j\in J} \to (l,\tau)$. Then, applying Corollary 2 to each legislative district, in equilibrium, maximizing the expected plurality or the expected probability of winning is equivalent to maximizing the Nash social welfare function for the district, subject to an additional constraint from the legislative rule, $\gamma(\cdot)$. The following corollary characterizes the equilibrium platform in each district.

COROLLARY 3: *In the voting game in district i, the equilibrium platform satisfies*

$$l_i^*, \tau_i^* \in \operatorname{argmax} \int_{\Omega_i} \ln u\left(\omega l_i(\omega) - \tau_i(\omega), l_i(\omega), b\int_{\Omega} \tau_i(\omega)\, dF(\omega)\right) dF_i(\omega)$$

$$\text{s.t.} \quad \omega l_i'(\omega) - \tau_i'(\omega) + u_2\, l_i'(\omega) = 0, \qquad (\text{IC}_i)$$

$$l_i'(\omega) > 0, \qquad (\text{M}_i)$$

$$\gamma(\cdot):\{(l_j,\tau_j)\}_{j\in J} \to (l,\tau), \qquad (\text{LEG}_i)$$

and the four boundary constraints, (B1)–(B4), hold.

The second stage is a legislative game. We consider a generalized version of the Baron–Ferejohn (1989) *random recognition* rule, which is a stylized version of the closed rule used in the U.S. House of Representatives. This version can be extended to incorporate many other different processes, as explained later. At the beginning of period *t*, legislator *j* is recognized as a proposer with probability $p_j^t \in [0,1]$, $\sum_{j\in J} p_j^t = 1$, $\forall t$.

Whoever is recognized proposes a tax schedule, (l_j^t, τ_j^t), then every legislator votes yes or no simultaneously. If, under m-majority rule, the number who say "yes" is greater than or equal to m, then (l_j^t, τ_j^t) becomes the new status quo and the game ends; otherwise, the game proceeds to period $t + 1$. Define $U_i(l, \tau) = \int_{\Omega_i} \ln u(\omega l(\omega) - \tau(\omega), l(\omega), b\int_{\Omega} \tau(\omega)\, dF(\omega))\, dF_i(\omega)$. If nothing gets passed forever, the payoff to the legislators is zero: $U_j(\phi) = 0$, for all $j \in J$.

We model the legislative process as a *stochastic game,* $\Gamma^t = (S^t, \pi^t, \psi^t)$, where S^t is the set of *pure strategy n tuples,* where $\pi^t : S^t \rightarrow \mu(Z)$ is a *transition function* specifying for each $s^t \in S^t$ a probability distribution $\pi^t(s^t)$ on Z, the set of states that can be achieved in a game, and where $\psi^t : S^t \rightarrow X$ is an *outcome function* that specifies for each $s^t \in S^t$ an outcome $\psi^t(s^t) \in X$. Finally, we use $S = \prod_{t \in T} S^t$ to denote the collection of pure strategy n tuples, where $S^t = \prod_{i \in N} s_i^t$. Formally, $Z = R \cup P \cup V$ is the set of states. We use z to denote the possible states the game moves to. We use R to denote the *Recognition Game,* P to denote the *Proposal Game,* and V to denote the *Voting Game.*

To simplify the analysis, committee structure in the legislature is not explicitly modeled here. One way to incorporate it is to model the committee game as a separate bargaining game prior to the legislative bargaining game. Because of the mathematical complexity involved, we do not incorporate it in this model. The U.S. Senate has increasingly resorted to unanimity rule, which can be incorporated into the game by requiring $m = |J|$. By varying p_j^t we can also incorporate other rules, such as the sequential recognition rule and seniority rule. In all these variations, Proposition 3 still holds. In practice, one can think of some modified rules that cannot be incorporated in the generalized version of the Baron–Ferejohn rule, but we argue that it is general enough to capture the main features of many legislative processes, and yet still simple enough to give us a handle to model it rigorously.

In the legislative game, each legislator's objective function is to maximize the Nash social welfare function of his district, $U_i(l, \tau)$, subject to the incentive compatibility constraint, the monotonicity condition, and the legislative rules, as stated in Corollary 3.

As usual, there are an infinite number of equilibria to the stochastic game. In what follows, we characterize the simplest equilibria involving no stage-dominated strategies, as noted by Baron and Kalai (1993): "We assume that when the group faces a set of equilibria which are all satisfactory from considerations of stability and efficiency, their attention is likely to be directed to a simple one".[4]

The simplest equilibrium can be described by an automaton of size 4, with one "rest" state (the Recognition Game), one "propose" state (the

[4]See Baron and Kalai (1993) for an analysis of the simplest equilibrium in the majority rule divide-the-dollar game with a random recognition rule.

Proposal Game), and the "vote yes" and "vote no" state (the Voting Game). The resulting equilibria from the automaton are stationary equilibria, which are each characterized by a set of values $\{v_t\} \subseteq \mathcal{R}^n$ for each stage of the game, and a strategy profile $\sigma^* \in \Sigma$, such that

(a) $\forall t$, σ^* is a Nash equilibrium with payoff function $G^t : \Sigma^t \to R^n$ defined by

$$G^t(\sigma^t;v) = U(\psi^t(\sigma^t)) + \sum_{z\in Z} \pi^t(\sigma^t)(z)v^z$$

$$= E_{\sigma^t}\left[U(\psi^t(s^t)) + \sum_{z\in Z} \pi^t(s^t)(z)v^z \right]$$

$$= \sum_{s^t\in S^t} \sigma^t(s^t)\left[U(\psi^t(s^t)) + \sum_{z\in Z} \pi^t(s^t)(z)v^z \right].$$

(b) $\forall t$, $v^t = U^t(\sigma^t;v)$.

We use the average payoff for each legislator's payoff for the entire stochastic game. So a legislator's payoff for the entire game is

$$U_i(\{I^t,\tau^t\}_t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} U_i^t(\sigma^t;v).$$

Let $\bar{U}_i = \sum_{j\in J} p_j U_i(l_j,\tau_j)$ represent the expected payoffs to player i at the beginning of each stage game. In the following proposition, we prove that the equilibrium strategy for legislator j is to vote yes with probability 1 if $U_j(l_i,\tau_i) \geq \bar{U}_j$, and to vote no otherwise. The equilibrium strategy for any proposer is to maximize his own utility such that the "least expensive" $m - 1$ members of the legislature would vote yes. Denote the set of legislators whose payoffs from the proposed tax schedule are greater than or equal to their continuation value as $M = \{k \in J\colon U_k(l_j,\tau_j) \geq \bar{U}_k\}$. Therefore, in equilibrium, proposer i proposes the tax schedule (l_i,τ_i) that maximizes the Nash social welfare of his own district subject to the constraint that at least $m - 1$ other players also vote yes, and his proposal will be accepted. Baron (1993) characterizes similar equilibrium strategies with alternatives in the Euclidean space and presents a closed-form characterization of the equilibrium when the utility function is quadratic. Proposition 3 is a generalization of Baron's results when the set of alternatives lies in a function space with generalized utility functions.

Some more notations will be used in solving for the equilibrium tax schedule for the legislative game. Let $\Omega_i \subset \Omega$ be the set of voters in legislator i's district. We will use the indicator function

$$\chi_i(\omega) = \begin{cases} 1 & \text{if } \omega \in \Omega_i, \\ 0 & \text{if } \omega \in \Omega\backslash\Omega_i. \end{cases}$$

PROPOSITION 3: *The following is a simplest subgame perfect stationary Nash equilibrium to the legislative game with stage-undominated strategies:*

For $z \in P$ and $i = p$ (Proposer i):

$$l_i, \tau_i \in \operatorname{argmax} \int_\Omega \chi_i(\omega) \ln u\left(\omega l_i(\omega) - \tau_i(\omega), l_i(\omega), b \int_\Omega \tau_i(\omega)\, dF(\omega) \right) dF_i(\omega)$$

$$\text{s.t.} \quad \omega l_i'(\omega) - \tau_i'(\omega) + u_2\, l_i'(\omega) = 0,$$

$$l_i'(\omega) > 0,$$

$$|\{k \in J \backslash \{i\} : U_k(l_i, \tau_i) \geq \bar{U}_k\}| \geq m - 1,$$

and the four boundary constraints, (B1)–(B4), hold.

For $z \in V$ and $j \in J \backslash \{i\}$ (Voter j):

$$s_j(l_i, \tau_i) = \begin{cases} 1 & \text{if } U_j(l_i, \tau_i) \geq \bar{U}_j, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Proposition 3 is given in Appendix B.

We use a three-district example to solve the stationary equilibrium tax schedule for the legislative game. It can be easily extended to the J-district case. In the following legislative game, $J = 3$, $m = 2$, and $p_i = 1/3$, for $i = 1,2,3$. The problem in Proposition 3 reduces to

$$\max_{\tau_i, l_i} \int_\Omega \chi_i(\omega) \ln u\left(\omega l_i(\omega) - \tau_i(\omega), l_i(\omega), b \int_\Omega \tau_i(\omega)\, dF(\omega) \right) dF_i(\omega)$$

$$\text{s.t.} \quad \omega l_i'(\omega) - \tau_i'(\omega) + u_2\, l_i'(\omega) = 0, \tag{IC$_i$}$$

$$l_i'(\omega) > 0, \tag{M$_i$}$$

$$U_j(l_i, \tau_i) \geq \bar{U}_j, \tag{LEG$_i$}$$

the four boundary constraints, (B1)–(B4), hold, and j is i's coalition member. Equation (LEG$_i$) can be expanded as

$$\int_\Omega \chi_j(\omega) \ln u(\omega l_i(\omega) - \tau_i(\omega), l(\omega), y_i)\, dF_j(\omega)$$

$$\geq \frac{1}{2} \int_\Omega \chi_j(\omega) [\ln u(\omega l_j(\omega) - \tau_j(\omega), l_j(\omega), y_j)$$

$$+ \ln u(\omega l_k(\omega) - \tau_k(\omega), l_k(\omega), y_k)]\, dF_j(\omega).$$

Apart from the usual incentive compatibility and monotonicity constraints, the tax schedule has to pass a majority of the legislature. The last constraint, (LEG$_i$), requires the payoff to the other member of the majority coalition, legislator j, to be at least his continuation value.

PROPOSITION 4:

(a) *The equilibrium tax schedule,* (l_i, τ_i)*, for the legislative game under a random recognition rule in the three-district case, satisfies* (IC_i), (M_i), LEG_i*, and the following equation:*

$$T_i' = \left\{ \int_{\underline{\omega}}^{\omega} [\chi_i(\omega) f_i(\omega) + \lambda(\omega)\chi_j(\omega) f_j(\omega)]/u \, dF - bF(\omega) \right.$$

$$\left. \times \int_{\Omega} [\chi_i(\omega) f_i(\omega) + \lambda(\omega)\chi_j(\omega) f_j(\omega)]\, u_3/u \, dF(\omega) \right\}$$

$$\times \frac{(1 + u_{22}\, l')}{wbf \int_{\Omega} [\chi_i(\omega) f_i(\omega) + \lambda(\omega)\chi_j(\omega) f_j(\omega)]\, u_3/u \, dF(\omega)},$$

where $\lambda(\omega) \geq 0$*, i is the proposer, and j is the legislator in the majority coalition with i.*

(b) *It is optimal if the welfare function is*

$$\int_{\Omega} A(u)\, dF(\omega) = \int_{\Omega} [\chi_i(\omega) f_i(\omega) + \lambda(\omega)\chi_j(\omega) f_j(\omega)] \ln u \, d\omega.$$

Proof: Define the function J as

$$J = \int_{\Omega} \{[\chi_i(\omega) f_i(\omega) + \lambda(\omega)\chi_j(\omega) f_j(\omega)] \ln u$$

$$+ \xi(\omega)[\omega l'(\omega) - \tau'(\omega) + u_2\, l_i'(\omega)]\}\, d\omega.$$

Let $g(l_i, \tau_i, l_{-i}, \tau_{-i}) = \int_{\Omega} \chi_j(\omega) \ln u(\omega l_i(\omega) - \tau_i(\omega), l(\omega), y_i)\, dF_j(\omega) - \frac{1}{2}\int_{\Omega} \chi_j(\omega)[\ln u(\omega l_j(\omega) - \tau_j(\omega), l_j(\omega), y_j) + \ln u(\omega l_k(\omega) - \tau_k(\omega), l_k(\omega), y_k)]\, dF_j(\omega)$. The complementary slackness condition requires

$$\lambda(\omega) g(l_i, \tau_i, l_{-i}, \tau_{-i}) = 0, \quad \text{with } \lambda(\omega) \geq 0.$$

The rest of the proof is similar to that of Proposition 1, with $A(u) = [\chi_i(\omega) f_i(\omega) + \lambda(\omega)\chi_j(\omega) f_j(\omega)] \ln u$. ■

Notice that the equilibrium tax schedule of the legislative process is different from that of the two-candidate competition. The difference comes from the specific forms of the social welfare functions. Therefore, individuals from districts whose legislators are not in the majority coalition get zero weight, while individuals whose legislators are in the majority coalition get positive weights in the social welfare function. This confirms our conjecture that the welfare weights of the optimal income tax schedule are endogenously determined by the political processes.

One special case is when all districts are identical, for then the single district case has the same policy outcome as the multiple district case.

When the districts are heterogeneous, however, the outcome of the legislative process in multiple districts will usually be different from that of a single district. Although analytical solutions and further comparative statics results are difficult to obtain without more specific assumptions on preferences and distributions of wage rates, we can compute the explicit equilibrium income tax schedules and public goods levels numerically once we parameterize the utility functions and distribution of wage rates. Section 4 provides a numerical example.

4. An Example

We will use a simplified economy to show the difference in income tax functions under different political institutions. Suppose wage rate, ω, is uniformly distributed in the interval [1,4]. Individuals have quasilinear utility functions of the form $I - \tau + \ln(1 - I/\omega) + \ln y + e$, where y is the amount of public good produced and e is the initial endowment for everyone.

Under a two-party plurality system in a single district, in equilibrium both candidates maximize the Nash social welfare function of the whole district subject to the incentive compatibility constraint and the monotonicity constraint:

$$\max_{\tau(),\, l()} \int_1^4 \ln\left[\omega l - \tau + \ln(1 - l) + \ln\left(\int_1^4 \tau/3 d\omega\right) + e\right] d\omega$$

$$\text{s.t.} \quad \omega l' - \tau' = \frac{l'}{1 - l},$$

$$l' > 0.$$

There is no analytical solution to the above system, so we resort to numerical solutions. For simplicity of calculation, we normalize $\tau(1) = 0$ and let $e = 5.0$.

Figure 1 shows the income tax schedule, $T(I)$, which is an increasing function of income, with the marginal tax rate at the lowest end of income being zero. The level of public good provided is 0.2892.

It is instructive to compare the outcome of single district case with that of the multiple district case. We consider the case when there are three districts, each with a uniform distribution of wage rates over the intervals, [1,2), [2,3), and [3,4]. Then there are eight possible cases of legislative coalition formation. We use the symbol $\rightarrow$ to represent "propose to and form coalition with". The eight cases are

$(1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1)$, $(1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 2)$,
$(1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 1)$, $(1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 2)$,
$(1 \rightarrow 3, 2 \rightarrow 3, 3 \rightarrow 1)$, $(1 \rightarrow 3, 2 \rightarrow 3, 3 \rightarrow 2)$,
$(1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 1)$, $(1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2)$.

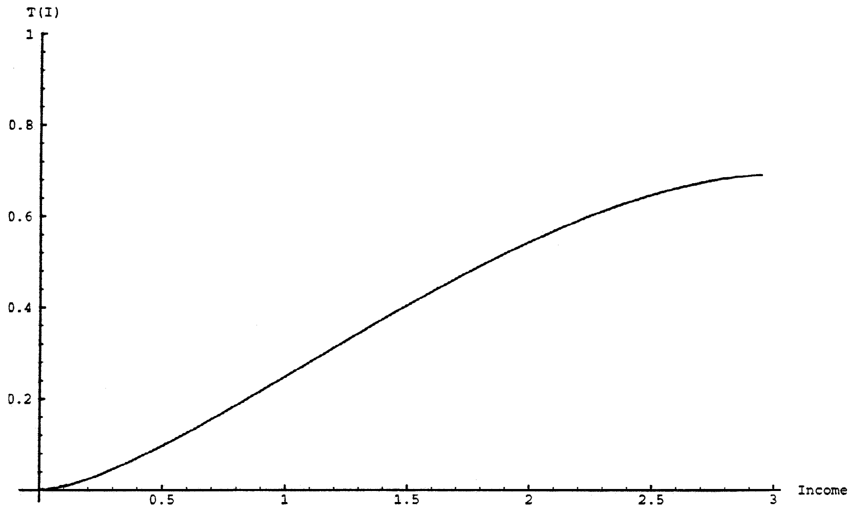

Figure 1: Income tax function: Single district.

As an example, the first case is set up below.

$$\max_{\tau_1, l_1} \int_1^2 \ln\left[\omega l_1 - \tau_1 + \ln(1 - l_1) + \ln\left(\int_1^4 \tau_1/3\, d\omega\right) + e\right] d\omega$$

$$\text{s.t.} \quad \omega l_1' - \tau_1' = \frac{l_1'}{1 - l_1},$$

$$l_1' > 0,$$

$$\int_2^3 \ln\left[\omega l_1 - \tau_1 + \ln(1 - l_1) + \ln\left(\int_1^4 \tau_1/3\, d\omega\right) + e\right] d\omega$$

$$\geq \frac{1}{2}\left\{\int_2^3 \ln\left[\omega l_2 - \tau_2 + \ln(1 - l_2) + \ln\left(\int_1^4 \tau_2/3\, d\omega\right) + e\right] d\omega\right.$$

$$\left. + \int_2^3 \ln\left[\omega l_3 - \tau_3 + \ln(1 - l_3) + \ln\left(\int_1^4 \tau_3/3\, d\omega\right) + e\right] d\omega\right\};$$

$$\max_{\tau_2, l_2} \int_2^3 \ln\left[\omega l_2 - \tau_2 + \ln(1 - l_2) + \ln\left(\int_1^4 \tau_2/3\, d\omega\right) + e\right] d\omega$$

$$\text{s.t.} \quad \omega l_2' - \tau_2' = \frac{l_2'}{1 - l_2},$$

$$l_2' > 0,$$

$$\int_3^4 \ln\left[\omega l_2 - \tau_2 + \ln(1 - l_2) + \ln\left(\int_1^4 \tau_2/3\, d\omega\right) + e\right] d\omega$$
$$\geq \frac{1}{2}\left\{\int_3^4 \ln\left[\omega l_3 - \tau_3 + \ln(1 - l_3) + \ln\left(\int_1^4 \tau_3/3\, d\omega\right) + e\right] d\omega\right.$$
$$\left. + \int_3^4 \ln\left[\omega l_1 - \tau_1 + \ln(1 - l_1) + \ln\left(\int_1^4 \tau_1/3\, d\omega\right) + e\right] d\omega\right\};$$

$$\max_{\tau_3,\, l_3} \int_3^4 \ln[\omega l_3 - \tau_3 + \ln(1 - l_3) + \ln\left(\int_1^4 \tau_3/3 d\omega\right) + e] d\omega$$

$$\text{s.t.} \quad \omega l_3' - \tau_3' = \frac{l_3'}{1 - l_3},$$

$$l_3' > 0,$$

$$\int_1^2 \ln\left[\omega l_3 - \tau_3 + \ln(1 - l_3) + \ln\left(\int_1^4 \tau_3/3\, d\omega\right) + e\right] d\omega$$
$$\geq \frac{1}{2}\left\{\int_1^2 \ln\left[\omega l_1 - \tau_1 + \ln(1 - l_1) + \ln\left(\int_1^4 \tau_1/3\, d\omega\right) + e\right] d\omega\right.$$
$$\left. + \int_1^2 \ln\left[\omega l_2 - \tau_2 + \ln(1 - l_2) + \ln\left(\int_1^4 \tau_2/3\, d\omega\right)\right] d\omega\right\}.$$

The equilibrium proposals of all three legislators can be calculated using numerical solutions. Figure 2 shows the equilibrium income tax functions proposed by the three legislators respectively. T_{1i} is the income tax function proposed by the legislator from district i. The public goods levels as outcomes of the three proposals are 0.3667, 0.2892, and 0.3272 respectively. It is interesting to observe that the equilibrium proposal of Legislator 2, the representative of the "middle productivity" district, coincides with the equilibrium proposal of the single district case. This example also points out some weakness of the empirical studies. The legislative outcome depends crucially on which legislator is the proposer. Therefore, without examining the exact legislative procedure used one cannot generalize as to whether popular vote (corresponding to the single district case) or legislative process tends to generate larger government spending.

5. Conclusions

This paper has three contributions to the literature of voting and optimal income taxation. First, it extends the probabilistic voting model to a function space. Second, it extends the Baron–Ferejohn (1989) legislative bargaining model to a function space. Third, it endogenizes the social

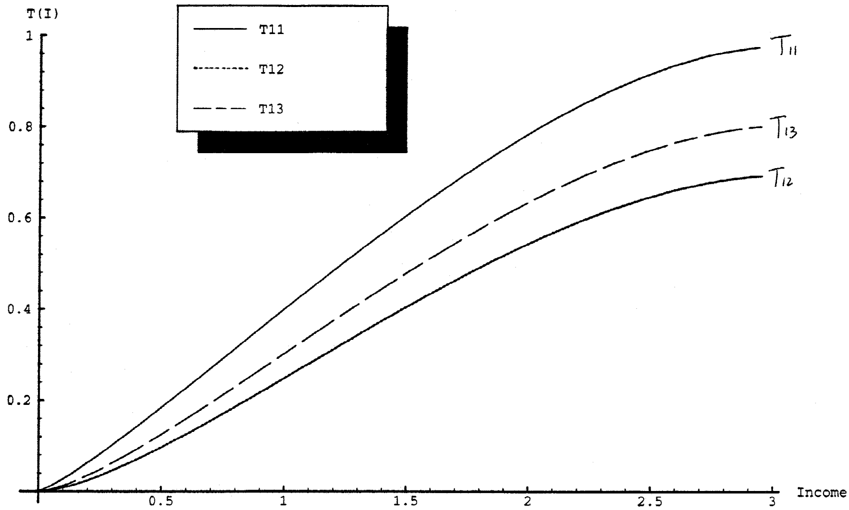

Figure 2: Income tax function: Three districts.

welfare functions in the normative optimal taxation literature by explicitly modeling different political institutions. Therefore, it provides a theoretical underpinning for viewing the prescriptions of normative economics as predictions about policy choices in different political equilibria.

We show that under a two-party plurality system with a single district the equilibrium income tax is equivalent to an optimal tax schedule that puts equal welfare weight over the whole population; when there are multiple districts, however, the simplest subgame perfect stationary equilibrium to the legislative game is equivalent to an optimal tax schedule that puts more welfare weight on the subsets of the population whose legislators are in the winning coalition of the legislature. Thus we have shown that the political processes endogenously determine the welfare weights of the optimal income taxation problem.

Appendix A

Proof of Lemma 1: We first derive the incentive compatibility constraint from the first approach (the indirect mechanism), where the government sets the income tax schedule, $T(I)$, and agent ω chooses labor supply, l, by maximizing the utility

$$\max_{l} u(\omega l - T(\omega l), l, y).$$

The first-order condition is

$$\omega - T'\omega + u_2 = 0. \tag{2}$$

The second approach uses the direct mechanism: The government sets the revenue requirement function, $\tau(\omega)$, and labor supply schedule, $l(\omega)$, then agent ω reports his type, ω', to maximize his utility

$$\max_{\omega'} u(\omega l(\omega') - \tau(\omega'), l(\omega'), y).$$

The first-order condition is

$$\frac{du}{d\omega'} = \omega \frac{dl(\omega')}{d\omega'} - \frac{d\tau(\omega')}{d\omega'} + u_2 \frac{dl(\omega')}{d\omega'} = 0.$$

Truthful revelation requires $\frac{du}{d\omega'}|_{\omega'=\omega} = 0$; that is,

$$\left.\frac{du}{d\omega'}\right|_{\omega'=\omega} = \omega \frac{dl(\omega)}{d\omega} - \frac{d\tau(\omega)}{d\omega} + u_2 \frac{dl(\omega)}{d\omega} = 0.$$

Using the shorthand $l'(\omega)$ to stand for $\frac{dl(\omega)}{d\omega}$, and similarly for other variables, the incentive compatibility constraint becomes

$$\omega - \frac{\tau'(\omega)}{l'(\omega)} + u_2 = 0. \tag{3}$$

Since $\tau(\omega') = T(\omega l(\omega'))$, differentiating with respect to ω' yields $\tau' = T'\omega l'$. Therefore, equations (2) and (3) are equivalent. ■

Proof of Lemma 2: From Assumption 1, $u(x, y, l)$ is C^3, and

$$u(x, y, l) = \nu(l, y) + \omega l - \tau \equiv V(\omega, l, y) - \tau.$$

Therefore the Spence–Mirrlees Condition is satisfied:

$$\frac{\partial^2 V}{\partial\omega\partial l}(\omega, l) = \frac{\partial}{\partial\omega}(\omega + \nu_l) = 1 > 0.$$

From Proposition 1 of Rochet (1987), if $l(\cdot)$ is increasing then $l(\cdot)$ is rationalizable; that is, there exists a transfer scheme $\tau(\cdot)$ such that $(l(\cdot), then \tau(\cdot))$ is truthfully implementable in dominant strategies. ■

Proof of Proposition 1: This is a calculus of variations problem. First we will present the transversality conditions, then solve for the first-order conditions with respect to τ and l, and last check the second-order conditions.

Define the function G as

$$G = A\left[u(\omega l(\omega) - \tau(\omega), l(\omega), b\int_\Omega \tau(\omega)\, dF(\omega)\right]$$
$$\times f(\omega) + \xi(\omega)[\omega l'(\omega) - \tau'(\omega) + u_2\, l'(\omega)].$$

Define the function J as

$$J = \int_{\Omega} G \, d\omega.$$

Step 1. Transversality conditions: Constraints (B1)–(B4) indicate that we need to solve for the transversality conditions with vertical starting and terminal lines. By a general result in Chiang (1992, pp. 61–64), we have

$$[G_{\tau'}]_{\omega=\underline{\omega}} = [G_{\tau'}]_{\omega=\bar{\omega}} = 0,$$

and

$$[G_{l'}]_{\omega=\underline{\omega}} = [G_{l'}]_{\omega=\bar{\omega}} = 0.$$

Since $G_{\tau'} = -\xi(\omega)$, it follows that

$$\xi(\underline{\omega}) = \xi(\bar{\omega}) = 0.$$

Next, $G_{l'} = \xi(\omega)[\omega + u_2]$. Evaluating this equation at the two boundaries does not give us more information about the system.

Step 2. First-order condition with respect to τ: Note that we cannot apply the Euler equation directly since there is an integral of τ in the G function. In what follows, we rederive the Euler equation for this nonstandard case.

$$\begin{aligned}
\delta J(\tau, h) &= \frac{d}{d\epsilon} J(\tau + \epsilon h)|_{\epsilon=0} \\
&= \int_{\Omega} \left\{ A' \left[-h + u_3 b \int_{\Omega} hf(\omega) \, d\omega \right] f(\omega) + \xi(\omega)(-h') \right\} d\omega \\
&= \int_{\Omega} [-A' f(\omega)] h \, d\omega + \left[\int_{\Omega} A' u_3 bf(\omega) \, d\omega \right] \\
&\quad \times \int_{\Omega} [f(\omega)] h \, d\omega - \int_{\Omega} \xi(\omega) h' \, d\omega \\
&= \int_{\Omega} \left\{ \left[-A' + \int_{\Omega} bA' u_3 f(\omega) \, d\omega \right] f(\omega) + \xi' \right\} h \, d\omega \\
&= 0, \quad \text{for all } h.
\end{aligned}$$

Note that we used integration by parts and the transversality conditions from the third to the fourth equality. It follows that

$$\left[-A' + \int_{\Omega} bA' u_3 f(\omega) \, d\omega \right] f(\omega) + \xi' = 0,$$

or

$$\xi' = A'f - bf\int_{\Omega} A'u_3\, dF(\omega). \tag{4}$$

Integrating both sides, we get

$$\xi(\omega) = \int_{\underline{\omega}}^{\omega} A'\, dF - bF(\omega)\int_{\Omega} A'u_3\, dF(\omega) + c,$$

where c is a constant. We can use the transversality condition to determine c:

$$\xi(\underline{\omega}) = 0 + c = 0.$$

Therefore,

$$\xi(\omega) = \int_{\underline{\omega}}^{\omega} A'\, dF - bF(\omega)\int_{\Omega} A'u_3\, dF(\omega). \tag{5}$$

Step 3. First-order condition with respect to l: Since this is the standard case, we can use the Euler equation for l directly:

$$\frac{\partial G}{\partial l} = \frac{d\left(\frac{\partial G}{\partial l'}\right)}{d\omega},$$

$$A'[\omega + u_2]f(\omega) = \xi'(\omega)[\omega + u_2] + \xi(\omega)[1 + u_{22}l'],$$

$$(A'f - \xi'(\omega))(\omega + u_2) = \xi(\omega)[1 + u_{22}l'].$$

From the inverse function theorem, we have $T' = \frac{\tau'}{I'} = 1 + \frac{u_2}{\omega}$. Plugging in equations (4) and (5) from Step 2, we get

$$T' = \frac{\left[\int_{\underline{\omega}}^{\omega} A'\, dF - bF(\omega)\int_{\Omega} A'u_3\, dF(\omega)\right](1 + u_{22}l')}{wbf\int_{\Omega} A'u_3\, dF(\omega)}. \tag{6}$$

Notice that T is also on the right-hand side of equation (6). Equations (6) and (IC) are the necessary conditions for a solution of the optimal income tax problem.

Step 4. Second-order conditions: To prove sufficiency, we need to check the concavity of G. Since G is linear in l' and τ', the Legendre and Weierstrass conditions are trivially satisfied. We only need to check the concavity of G in l and τ, which requires the matrix of the second partial derivatives with respect to l and τ to be negative semidefinite. Since both $A(\cdot)$ and u are concave in l and τ, we can decompose the matrix as a sum of two matrices where one of them is negative definite. Then the sufficient conditions are verified if the other matrix, derived from the incentive compatibility constraint, is concave in l and τ. Using Assumption 1 that u is C^3, we require the matrix

$$D = l' \begin{pmatrix} u_{222} & u_{223}\, b \\ u_{232}\, b & u_{233}\, b^2 \end{pmatrix}$$

to be negative semidefinite.

Constraint (M) gives us $l' > 0$. With additive separability, the sufficiency condition is reduced to requiring $u_{222} \leq 0$, which is satisfied from Assumption 2. Thus, the first-order condition characterized by equation (6) is also sufficient. ■

Proof of Lemma 3: An individual's after-tax consumption is $x(\omega,\omega_t) = \omega_t l(\omega) - \tau(\omega)$. He reports an optimal ω such that

$$x(\omega,\omega_t) + \nu(l(\omega), y) \geq x(\omega',\omega_t) + \nu(l(\omega'), y), \quad \forall \omega' \in \Omega.$$

Truthful revelation requires that the above inequality hold for $\omega = \omega_t$:

$$x(\omega_t,\omega_t) + \nu(l(\omega_t), y) \geq x(\omega',\omega_t) + \nu(l(\omega'), y), \quad \forall \omega' \in \Omega.$$

That is,

$$x(\omega_t,\omega_t) - x(\omega',\omega_t) \geq \nu(l(\omega'), y) - \nu(l(\omega_t), y).$$

If $\omega_t \geq \omega'$, by Lemma 2 we have $l(\omega_t) \geq x(\omega')$ in equilibrium, and, therefore, $\nu(l(\omega'), y) - \nu(l(\omega_t), y) \geq 0$. So $x(\omega_t,\omega_t) - x(\omega',\omega_t) \geq 0$. ■

Proof of Lemma 4: $\tau(\omega) = I(\omega) - x(\omega,\omega_t)$. From Lemmas 2 and 3 we know that in equilibrium both $I(\omega)$ and $x(\omega,\omega_t)$ are nondecreasing in ω. To see that both functions are bounded, recall that $I \in [0,\bar{\omega})$ and $x \in (0,\bar{I} - \underline{\tau}]$, where $\bar{I} - \underline{\tau} = \bar{\omega} + \int_\Omega \omega dF(\omega)$ by Assumption 3. By Jordan's Theorem (e.g., see Wheeden and Zygmund 1977), $\tau(\omega)$ is of bounded variation. ■

Proof of Lemma 5: Since $l \in [0,1)$, and l is increasing, l is of bounded variation and is variation norm bounded.

From Lemma 4, τ is of bounded variation. From Assumption 3,

$$\omega > \tau(\omega) > -\infty.$$

Therefore, τ is also variation norm bounded.

Let $M[a,b]$ be the set of all countably additive signed Borel measures on $[a,b]$. From Border (1991, Thm. 4.1), the $\sigma(BV, M)$ topology and the topology of pointwise convergence coincide on the set $\{(l \in X_l, \tau \in X_\tau)\}$.

Next, we show that adding the incentive compatibility constraint does not change pointwise convergence. The incentive compatibility constraint says:

$$\omega l_n(\omega) - \tau_n(\omega) + \nu(l_n(\omega), y)$$
$$\geq \omega l_n(\omega') - \tau_n(\omega') + \nu(l_n(\omega'), y), \quad \forall \omega' \in \Omega.$$

As $l_n(\omega) \to l(\omega)$ and $\tau_n(\omega) \to \tau(\omega)$, we have

$$\omega l(\omega) - \tau(\omega) + \nu(l(\omega), y) \geq \omega l(\omega') - \tau(\omega') + \nu(l(\omega'), y), \quad \forall \omega' \in \Omega.$$

So X is variation norm bounded and is a pointwise closed subset of bounded variation, and, therefore, from Border (1991, Thm. 4.1) is $\sigma(BV, M)$-compact. ■

Proof of Corollary 2: Let

$$EPl_i = \int_\Omega \frac{u(\omega l_i - \tau_i, l_i, y_i) - u(\omega l_{-i} - \tau_{-i}, l_{-i}, y_{-i})}{u(\omega l_i - \tau_i, l_i, y_i) + u(\omega l_{-i} - \tau_{-i}, l_{-i}, y_{-i})}\, d\omega.$$

Since $u(\omega l - \tau, l, y)$ is concave in (l,τ), it follows that the constrained objective function

$$O_i = EPl_i + \int_\Omega \xi(\omega)[\omega l'(\omega) - \tau'(\omega) + u_2\, l'(\omega)]\, d\omega$$

is concave in (l_i,τ_i), convex in (l_{-i},τ_{-i}), and continuous in both (l_i,τ_i) and (l_{-i},τ_{-i}). Note that Assumption 2 ensures that u_2 is concave in l. From Lemma 5, X is compact. Therefore, an electoral equilibrium exists.

Next, we show that $(l,\tau) \in X$ is an equilibrium to the electoral game if and only if it is a global maximum of $O_i((l_i,\tau_i),(l,\tau))$, given $(l_{-i},\tau_{-i}) = (l,\tau)$. This follows from the interchangeability condition for two-person, zero-sum games.

Let $W(l,\tau) = \int_\Omega \ln u(\omega l - \tau, l, y)\, dF(\omega)$. We then show that $l^*,\tau^* \in$ argmax $W(l,\tau)$ is equivalent to $l^*,\tau^* \in$ *argmax* $EPl_i((l_i,\tau_i),(l,\tau))$, for $i =$ 1,2. Since ln $u(\omega l - \tau, l, y)$ is a strictly monotone increasing concave function of $u(\omega l - \tau, l, y)$, then $W(l,\tau)$ is concave in (l,τ). Therefore, every local maximum of $W(l,\tau)$ is also a global maximum. Similarly, since $EPl_i((l_i,\tau_i),(l,\tau))$ is concave in (l_i,τ_i), it follows that any of its local maxima is also a global maximum. So the first-order conditions for the maximization problems are both necessary and sufficient. It suffices to show that the first-order conditions of the two functions are equivalent. We prove this by using calculus of variations.

$$W(\tau + \epsilon h) = \int_\Omega \ln u\left(\omega l - \tau - \epsilon h, l, \int_\Omega (\tau + \epsilon h)\, dF \right) dF(\omega).$$

Then,

$$\begin{aligned} \delta W(\tau; h) &= \frac{d}{d\epsilon} W(\tau + \epsilon h)|_{\epsilon=0} \\ &= \int_\Omega \left[-\frac{1}{u} + \int_\Omega \frac{u_3}{u}\, dF \right] h\, dF(\omega) \\ &= 0, \quad \text{for all } h. \end{aligned}$$

Therefore,

$$-\frac{1}{u} + \int_\Omega \frac{u_3}{u}\, dF = 0.$$

Similarly,

$$\begin{aligned} \delta EPl_1(\tau_1; h)|_{\tau_1 = \tau; I_1 = I} &= \frac{d}{d\epsilon} EPl_1(\tau + \epsilon h)|_{\epsilon=0; \tau_1 = \tau; I_1 = I} \\ &= \int_\Omega \frac{2u\left[-h + u_3 \int_\Omega h\, dF \right]}{(2u)^2} \\ &= \int_\Omega \left[-\frac{1}{2u} + \int_\Omega \frac{u_3}{2u}\, dF \right] h\, dF(\omega) \\ &= 0, \quad \text{for all } h. \end{aligned}$$

Therefore,

$$-\frac{1}{u} + \int_\Omega \frac{u_3}{u}\, dF = 0.$$

It follows that

$$\delta W(\tau; h) = 2 \cdot \delta EPl_1(\tau_1; h)|_{\tau_1 = \tau; l_1 = l},$$

so that $\delta W(\tau; h) \leq 0$ if and only if $\delta EPl_1(\tau_1; h)|_{\tau_1 = \tau; l_1 = l} \leq 0$. Similarly, we can prove the equivalence of first-order conditions with respect to *l*. Since the techniques are identical, we omit this part. ■

Appendix B

Proof of Proposition 3: We start by defining the strategy sets, transition functions, and outcome functions for the game elements:

$$\begin{array}{lll} \text{For } z \in R: & S_i^t = \{0\}, & \forall i \in J, \\ \text{(Recognition Game)} & \pi^t(s^t)(z) = 1, & \text{if } z \in P \\ & \psi^t(s^t) = \phi, & \forall s^t \in S^t. \end{array}$$

The Recognition Game is indexed by $z \in R$. The order of recognition is randomly decided according to some exogenously given probabilities; therefore, the strategy set of each player is $\{0\}$. The game proceeds to the Proposal Game with probability 1, and the null outcome prevails.

$$\begin{array}{lll} \text{For } z \in P: & S_i^t = \begin{cases} \{l_i, \tau_i\} & \text{if } i = p, \\ \{0\} & \text{if } i \in N \backslash \{p\}, \end{cases} \\ \text{(Proposal Game)} & \pi^t(s^t)(z) = 1, & \text{if } z \in V, \\ & \psi^t(s^t) = \phi, & \forall s^t \in S^t. \end{array}$$

In the Proposal Game, we use p to denote the Proposer. The strategy set for the Proposer is the set of tax schedules $\{l_i, \tau_i\}$, while the strategy set for each voter is still $\{0\}$. The game proceeds to the Voting Game with probability 1, and the null outcome prevails in this game.

$$\begin{array}{lll} \text{For } z \in V: & S_i^t = \{0,1\}, & \forall i \in J, \\ \text{(Voting Game)} & \pi^t(s^t)(z) = 1, & \text{if } z \in R, \\ & \psi^t(s^t) = \begin{cases} l_i^t, \tau_i^t & \text{if } \sum_{i \in J} S_i^t \geq m, \\ \phi & \text{otherwise.} \end{cases} & \forall s^t \in S^t. \end{array}$$

In the Voting Game, each player can vote either no or yes (0 or 1) to the proposed tax schedule. If the new proposal, (l_i, τ_i), is accepted by at least m of the legislators, it becomes the new status quo; otherwise, the null outcome prevails for this period and the game moves to a new round starting from the Recognition game with probability 1.

The main steps to prove Proposition 3 follow the definition of stationary Nash equilibrium. We first specify the values associated with the equilibrium strategies, and then show that these values are self-generating. The third step is to show that the strategies specified in the proposition are subgame perfect Nash equilibria.

The values of the games are defined below. The interpretations of these values go back to the definitions of each game element above.

For $z \in R$: $v_i^t = v_i^{(R)}$, $\forall i \in R$.

(Recognition Game)

For $z \in P$: $v_i^t(l_i, \tau_i) = U_i(l_i, \tau_i)$, for $i = p$,

(Proposal Game) $v_j^t(l_i, \tau_i) = \bar{U}_j$, for $j \in J \setminus \{p\}$,

where

$$l_i, \tau_i \in \text{argmax} \int_\Omega \chi_i(\omega) \ln u(\omega l_i(\omega) - \tau_i(\omega), l_i(\omega), y_i)\, dF_i(\omega)$$

s.t. $(\text{IC}_i), (\text{M}_i), (\text{LEG}_i), (\text{B1}), (\text{B2}), (\text{B3}), (\text{B4})$.

For $z \in V$: $v_i^t = \alpha(|M|)\, U_j(l_i, \tau_i) + (1 - \alpha(|M|))\, \bar{U}_j(l_i, \tau_i)$,

(Voting Game) $\forall j \in J$,

where

$$\alpha(|M|) = \begin{cases} 1 & \text{if } |M| \geq m, \\ 0 & \text{otherwise.} \end{cases}$$

The next step is to verify that these values are self-generating – that they correspond to the payoffs under the equilibrium strategies. To do this, we plug the equilibrium strategies and other game elements into the definition of G, and show that they equal the corresponding values.

For $z \in R$ (Recognition Game):

$$G^t(\sigma^t, v^t) = E_{\sigma^t}\left[U(\psi^t(s^t)) + \sum_{z \in Z} \pi^t(s^t)(z) v^z \right]$$

$$= U(\phi) + \pi^t(s^t)(z) \cdot v^t$$

$$= v^{(R)} = v^t.$$

For $z \in P$ (Proposal Game):

For $i = p$ (Proposer i):

$$G_i^t(\sigma^t; v_i^t) = E_{\sigma^t}\left[U(\psi^t(s^t)) + \sum_{z \in Z} \pi^t(s^t)(z) v^z \right]$$

$$= U_i(\phi) + 1 \cdot U_i(l_i, \tau_i)$$

$$= U_i(l_i, \tau_i)$$

$$= v_i^t(l_i, \tau_i).$$

For $j = J \backslash \{p\}$ (Voter j):

$$G_j^t(\sigma^t; v_t^j) = E_{\sigma^t}\left[U(\psi^t(s^t)) + \sum_{z \in Z} \pi^t(s^t)(z) v^z \right]$$

$$= U_j(\phi) + 1 \cdot \bar{U}_j$$

$$= \bar{U}_j$$

$$= v_j^t(l_i, \tau_i).$$

For $z \in V$ (Voting Game):

$$G^t(\sigma^t, v^t) = E_{\sigma^t}\left[U(\psi^t(s^t)) + \sum_{z \in Z} \pi^t(s^t)(z) v^z \right]$$

$$= \alpha(|M|)\, U_j(l_i, \tau_i) + (1 - \alpha(|M|))(U_j(\phi) + \bar{U}_j(l_i, \tau_i))$$

$$= v^t.$$

Next, we verify that the strategies specified in Proposition 3 are subgame perfect Nash equilibrium strategies. Since the strategies are history-independent, it suffices to show that for each game element no player will benefit from a unilateral one-shot deviation.

For $z \in P$, we want to show that tax proposal (l_i, τ_i) is the equilibrium strategy for Proposer i, where

$$l_i, \tau_i \in \operatorname{argmax} \int_\Omega \chi_i(\omega) \ln u(\omega l_i(\omega) - \tau_i(\omega), l_i(\omega), y_i)\, dF_i(\omega)$$

$$\text{s.t.} \quad (\text{IC}_i), (\text{M}_i), (\text{LEG}_i), (\text{B1}), (\text{B2}), (\text{B3}), (\text{B4}).$$

The corresponding payoff for Proposer i is

$$G_i^t(\sigma^t; v^t(l_i, \tau_i)) = U_i(l_i, \tau_i).$$

If the proposer defects to any other pure strategy $(l_i^0, \tau_i^0) \neq (l_i, \tau_i)$, $\forall i \in J$, there are two possible consequences:

(i) $U_i(l_i^0, \tau_i^0) \leq U_i(l_i, \tau_i)$, in which case he is not better off by defection, so he will not defect.

(ii) $U_i(l_i^0, \tau_i^0) > U_i(l_i, \tau_i)$: in which case, if $|\{k \in J \backslash \{i\} : U_k(l_i^0, \tau_i^0) \geq \bar{U}_k\}| \geq m - 1$ still holds, $\forall j \neq i$, then

$$l_i, \tau_i \notin \text{argmax} \int_{\Omega} \chi_i(\omega) \ln u(\omega l_i(\omega) - \tau_i(\omega), l_i(\omega), y_i)\, dF_i(\omega)$$

$$\text{s.t.} \quad (\text{IC}_i), (\text{M}_i), (\text{LEG}_i), (\text{B1}), (\text{B2}), (\text{B3}), (\text{B4});$$

but this contradicts the definition of (l_i, τ_i).

So the proposer has no positive incentive to defect unilaterally from his strategy specified in Proposition 3, which means that it is a Nash equilibrium for the proposer. Since it is history independent, it is also a subgame perfect equilibrium.

For $z \in V$, we want to check if voters' strategies specified in the proposition are Nash equilibrium strategies. We consider three cases:

1. When $|M| > m$, no voter is pivotal, so voters have no positive incentive to defect from their equilibrium strategies.
2. When $|M| = m$, any voter $i \in M$ is pivotal. Since $G_i(s_i = 0, s_{-i}{}^*) - G_i(s_i = 1, s_{-i}{}^*) = \bar{U}_i - U_i(l_i, \tau_i) \leq 0$, i has no positive incentive to defect from his equilibrium strategy.
3. When $|M| = m - 1$, any voter $i \in J \backslash M$ is pivotal. Since $G_i(s_i = 1, s_{-i}{}^*) - G_i(s_i = 0, s_{-i}{}^*) = U_i(l_i, \tau_i) - \bar{U}_i \leq 0$, i has no positive incentive to defect either.

Therefore, the voter strategies specified in the proposition are Nash equilibrium strategies. They are subgame perfect, since they are history independent. ■

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